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One-parameter class of cosmological fluids with time dependent adiabatic indices for accelerating universes

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Abstract. - A one-parameter family of time dependent adiabatic indices is introduced for any given type of cosmological fluid of constant adiabatic index by a mathematical method belonging to the class of Darboux transformations. The procedure works for zero cosmological constant at the price of introducing a new constant parameter related to the time dependence of the adiabatic index. I argue that these are the real cosmological fluids that are encountered at cosmological scales and that they can provide a simple and efficient explanation for the recent experimental findings regarding the present day accelerating universe.

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1 - Recent data from type Ia supernovae that can still be considered as meager provoked tremendous interest since they indicate that the universe has *now* an accelerating large scale expansion [1] ($\rho + 3\mathrm{p} < 0$ in eq. (1) below). This supernovae revolution stimulated model builders to supply a remarkable amount of paper work on this new topic, in which such

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impressive words as *quintessence* can be found. On the other hand, simple explanations should always be taken into account on the background of ever more complicated modeling. In the following, I would like to act according to the latter rule and add a hopefully nontrivial contribution to this literature.

2 - The scale factor a(t) of a FLRW metric is a function of the comoving time t obeying the Einstein-Friedmann dynamical equations (EFDEs):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G\rho}{3} - \frac{\kappa}{a^2} \ , \tag{2}$$

where ρ and p are the energy density and the pressure, respectively, of the perfect fluid of which a classical universe is usually assumed to be made of, and $\kappa = 0, \pm 1$ is the curvature index of the flat, closed, open universe, respectively. To solve EFDEs for a(t) one needs an equation of state for the fluid, i.e., a relationship between p and ρ. Very used ones are the barotropic equations of state (BES)

$$p = (\gamma - 1)\rho, \tag{3}$$

where γ is the constant adiabatic index. As well known, all the important cosmological fluids assumed to be dominant along various cosmological epochs belong to the barotropic class. It is straightforward to get a(t) by direct integration for the flat case, however there are some difficulties in the other two cases. Landau and Lifschitz [2] gave a general procedure to perform the integration, which is valid for any γ. On the other hand, for the purposes of the present work, an equivalent procedure introduced by Faraoni [3] leading practically to the same results (however with key constraints) is very useful.

3 - The alternative method of Faraoni to derive the scale factor of the universe is based on the Riccati equation and goes as follows. First, one combines EFDEs and BES yielding

$$\frac{\ddot{a}}{a} + c\left(\frac{\dot{a}}{a}\right)^2 + \frac{c\kappa}{a^2} = 0 \ , \tag{4}$$

where $c = \frac{3}{2}\gamma - 1$. The case $\kappa = 0$ is directly integrable [3] and I shall not pay attention to it in the present work, although it can be included in the scheme. Eq. (4) is rewritten in the conformal time variable η in the form

$$\frac{\mathrm{a}''}{\mathrm{a}} + (\mathrm{c}-1)\left(\frac{\mathrm{a}'}{\mathrm{a}}\right)^2 + \mathrm{c}\kappa = 0\ . \tag{5}$$

One can immediately see that by means of the change of function $u = \frac{a'}{a}$ the following Riccati equation is obtained (c > 0 henceforth)

$$\mathrm{u}' + \mathrm{c}\mathrm{u}^2 + \kappa\mathrm{c} = 0\ . \tag{6}$$

Furthermore, employing $\mathrm{u} = \frac{1}{\mathrm{c}}\frac{\mathrm{w}'}{\mathrm{w}}$ one gets the very simple second order differential equation

$$\mathrm{w}'' + \kappa\mathrm{c}^2\mathrm{w} = 0\ . \tag{7a}$$

For $\kappa = 1$ the solution of the latter is $\mathrm{w}_1 = \mathrm{W}_1\cos(\mathrm{c}\eta + \mathrm{d})$, where d is an arbitrary phase, implying

$$\mathrm{a}_1(\eta) = \mathrm{A}_1[\cos(\mathrm{c}\eta + \mathrm{d})]^{1/\mathrm{c}}\ , \tag{8}$$

whereas for $\kappa = -1$ one gets $\mathrm{w}_{-1} = \mathrm{W}_{-1}\sinh(\mathrm{c}\eta)$ and therefore

$$\mathrm{a}_{-1}(\eta) = \mathrm{A}_{-1}[\sinh(\mathrm{c}\eta)]^{1/\mathrm{c}}\ , \tag{9}$$

where $\mathrm{W}_{\pm1}$ and $\mathrm{A}_{\pm1}$ are amplitude parameters. Eqs. (8) and (9) are the same solutions as in the standard procedure. According to Faraoni, the alternative method requires γ be a constant. However, as I will show in the following there is still some room in this method for time-dependent adiabatic indices.

4 - I will now apply the so-called strictly isospectral Darboux technique (SIDT[2]) to eq. (7a). This is a mathematical approach related to the factorizations of the second order linear differential equations that underlined supersymmetric quantum mechanics and the dynamical symmetry breaking ideas of Witten almost two decades ago [4]. There is a slight but important difference between Witten's approach and SIDT. While Witten's factorizations rely on the particular Riccati solution, the SIDT uses the general Riccati solution. This was first discussed by Mielnik in 1984 [5]. The difference with respect to Mielnik is that I will employ the SIDT scheme at

[2]In my previous works,I have also used the acronym DDGR for double Darboux general Riccati to denote the same scheme.)

fixed *zero* 'energy' (see also [6]). I refer the reader to my short survey of Darboux transformations for more details [7]. During the last few years, I have acquired some experience with SIDT extending its range of applications beyond supersymmetric quantum mechanics, to subjects such as Newtonian damping [8], Fokker-Planck equation [9], and quantum cosmology [10]. It is my aim here to apply this mathematical scheme to the dynamics of the scale factor of a classical universe.

First of all, despite c looks like a frequency for the function w, I shall interpret $\pm$c as Schrödinger 'potentials'. This is preferable to $\pm c^2$ for algebraic reasons coming out from the bunch of formulas that I present next. (Notice that we have a constant potential having a coupling constant equal to the potential itself!). Thus, equation (7a) will be considered as a Schrödinger equation at zero 'energy' (or more rigorously at zero factorization constant).

The point now is that the Riccati solution $u_p = \frac{1}{c}\frac{w^{'}}{w}$ mentioned above is only the particular solution, i.e., $u_{p,1} = -\tan c\eta$ and $u_{p,-1} = \tanh c\eta$ for $\kappa = \pm 1$, respectively. The particular Riccati solutions are closely related to the common factorizations of the Schrödinger equation. Indeed eq. (7a) can be written

$$w^{''} - c(-\kappa c)w = 0 \tag{7b}$$

and using eq. (6) one gets

$$\left(\frac{d}{d\eta} + cu_p\right)\left(\frac{d}{d\eta} - cu_p\right)w = w^{''} - c(u_p^{'} + cu_p^2)w = 0\ . \tag{7c}$$

To fix the ideas, we shall call eq. (7c) the bosonic equation. On the other hand, the supersymmetric partner (or fermionic) equation of eq. (7c) will be

$$\left(\frac{d}{d\eta} - cu_p\right)\left(\frac{d}{d\eta} + cu_p\right)w = w^{''} - c(-u_p^{'} + cu_p^2)w = w^{''} - c\cdot c_{\kappa,\mathrm{susy}}w = 0\ . \tag{7d}$$

Thus, one can write

$$c_{\kappa,\mathrm{susy}}(\eta) = -u_p^{'} + cu_p^2 = \begin{cases} c(1 + 2\tan c\eta) & \text{if } \kappa > 1 \\ c(-1 + 2\tanh c\eta) & \text{if } \kappa < 1 \end{cases}$$

for the supersymmetric partner adiabatic index.

To find the general Riccati solution, one is led to consider the following Riccati equation $cu_g^2 - \frac{du_g}{d\eta} = cu_p^2 - \frac{du_p}{d\eta}$, whose general solution can be written down as $u_g(\eta) = u_p(\eta) - \frac{1}{v(\eta)}$, where $v(\eta)$ is an unknown function.

Using this ansatz, one obtains for the function $\mathrm{v}(\eta)$ the following Bernoulli equation

$$\frac{\mathrm{dv}(\eta)}{\mathrm{d}\eta} + 2\mathrm{c}\,\mathrm{v}(\eta)\,\mathrm{u_p}(\eta) = \mathrm{c}, \tag{10}$$

that has the solution

$$\mathrm{v}(\eta) = \frac{\mathcal{I}_\kappa(\mathrm{c}\eta) + \lambda}{\mathrm{w}_\kappa^2(\mathrm{c}\eta)}, \tag{11}$$

where $\mathcal{I}_\kappa(\mathrm{c}\eta) = \int_0^{\mathrm{c}\eta} \mathrm{w}_\kappa^2(\mathrm{y})\,\mathrm{dy}$, if we think of a half line problem for which λ is a positive integration constant thereby considered as a free SIDT parameter.

Thus, $\mathrm{u_g}(\eta)$ can be written as follows

$$\mathrm{u_g}(\eta;\lambda) = \mathrm{u_p}(\eta) - \frac{1}{\mathrm{c}}\frac{\mathrm{d}}{\mathrm{d}\eta}\Big[\ln(\mathcal{I}_\kappa(c\eta) + \lambda)\Big] \tag{12a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\eta}\Big[\ln\left(\frac{\mathrm{w}(\mathrm{c}\eta)}{\mathcal{I}_{\mathrm{c}\kappa}(\eta) + \lambda}\right)^{\frac{1}{\mathrm{c}}}\Big]. \tag{12b}$$

The important result provided by the SIDT is the one-parameter family of adiabatic indices $\mathrm{c}_\kappa(\eta;\lambda)$

$$\mathrm{c}_\kappa(\eta;\lambda) = c\mathrm{u_g^2}(\eta;\lambda) + \frac{\mathrm{du_g}(\eta;\lambda)}{\mathrm{d}\eta} \tag{13a}$$

$$= -\kappa\mathrm{c} - \frac{2}{\mathrm{c}}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\Big[\ln(\mathcal{I}_\kappa(\mathrm{c}\eta) + \lambda)\Big] \tag{13b}$$

$$= -\kappa\mathrm{c} - \frac{4\mathrm{w}_\kappa(\eta)\mathrm{w}_\kappa'(\eta)}{\mathrm{c}(\mathcal{I}_\kappa(\mathrm{c}\eta) + \lambda)} + \frac{2\mathrm{w}_\kappa^4(\eta)}{\mathrm{c}(\mathcal{I}_\kappa(\mathrm{c}\eta) + \lambda)^2}\ . \tag{13c}$$

One can easily infer the following formula for the parametric adiabatic indices $\gamma_\kappa(\eta;\lambda)$

$$\gamma_\kappa(\eta;\lambda) = \gamma + \frac{8}{3\kappa(3\gamma - 2)}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\Big[\ln(\mathcal{I}_\kappa(\mathrm{c}\eta) + \lambda)\Big]\ , \tag{13d}$$

which I used for plotting. All $\mathrm{c}_\kappa(\eta;\lambda)$ have the same supersymmetric partner index $\mathrm{c}_{\kappa,\mathrm{susy}}(\eta)$ obtained by deleting the zero mode solution w_κ. They may be considered as intermediates between the initial constant index $\kappa\mathrm{c}$ and the supersymmetric partner index $\mathrm{c}_{\kappa,\mathrm{susy}}(\eta)$. From eq. (12$b$) one can infer the new parametric 'zero mode' solutions of the universe for the family of barotropic indices $\mathrm{c}_\kappa(\eta;\lambda)$ as follows

$$\mathrm{w}_\kappa(\eta;\lambda) = \frac{\mathrm{w}_\kappa(\eta)}{\mathcal{I}_\kappa(\eta) + \lambda} \Longrightarrow \mathrm{a}_\kappa(\eta,\lambda) = \left(\frac{\mathrm{w}_\kappa(\eta)}{\mathcal{I}_\kappa(\eta) + \lambda}\right)^{\frac{1}{\mathrm{c}}}\ . \tag{14}$$

A so-called double Darboux feature of the SIDT can be inferred by writing the parametric family in terms of their unique "fermionic" partner index

$$\mathrm{c}_{\kappa}(\eta;\lambda)=\mathrm{c}_{\kappa,\mathrm{susy}}(\eta)+\frac{2}{\mathrm{c}}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\ln\left(\frac{1}{\mathrm{w}_{\kappa}(\eta;\lambda)}\right), \tag{15}$$

which shows that the SIDT is of the inverse Darboux type [11]. This is very important because it allows a two-step (double Darboux) interpretation, namely, in the first step one goes to the fermionic system and in the second step one returns to a deformed bosonic system. In addition, this interpretation helps me to introduce another conjecture, on which I commented in a different context elsewhere [12]. In particle physics supersymmetric ideas forces one to introduce supersymmetric partners for each known particle. However, if one promotes SIDT at the level of a matter of principles, one can think of a scenario in which the supersymmetric partner system is unstable/unphysical and only the systems generated by means of the general Riccati solution are encountered in nature. Thus, in the cosmological context, I claim that only the time-dependent barotropic fluids of indices $c_{\kappa}(\eta;\lambda)$ make the real material content of the universe. At the cosmological scale, the fluids of constant γ occur in the limit $\lambda\rightarrow\infty$ and therefore they look more as academic cases in this context. SIDT introduces a new time scale in zero-Λ cosmology, which is given by the integration constant λ. In ordinary supersymmetric quantum mechanics, this constant is related to the contribution of the irregular zero mode entering the SIDT parametric zero modes. Extrapolating this fact to cosmology, one can say that this time constant gives the contribution of the 'unphysical' solution to the parametric modes given by eq. (14). In the common approach, the 'unphysical' solution is discarded by imposing initial conditions.

Another comment I would like to make is that from the SIDT standpoint I have worked at both fixed zero factorization 'energy' and fixed coupling constant c of the Schrödinger 'potential'. Thus, this is different from both standard quantum mechanical framework, working at fixed coupling constant but variable energy, and the Sturmian approach, which works at fixed energy but variable coupling constant (depending only on quantum numbers). From the strictly technical point of view, what I have introduced here is a supersymmetric class of solutions for the scale factor of the universe corresponding to a family of cosmological fluids connected to any given fluid of constant adiabatic index by the SIDT supersymmetric procedure at zero factorization constant. One can interpret the denominator in eq. (14) as an amplitude damping for these supersymmetric scale factors whose origin is the time dependence of the adiabatic index. Some plots for the matter-dominated universe ($\mathrm{c} = \frac{1}{2}$) and radiation-filled universe ($\mathrm{c} = 1$) are presented in Figs. 1a,b,c,d and 2a,b,c,d, respectively.

5 - In conclusion, a supersymmetric class of cosmological fluids has been introduced in this work possessing time-dependent adiabatic indices. Such fluids may provide a simple explanation for a currently accelerating universe and therefore they should be of considerable interest. In future modeling along the lines of this work the parameter λ can be either a constant fixed e.g., by some variational principles or can be made dependent on other astroparticle parameters. At the general level, a richer dynamics of the scale factor of the universe is introduced by simple mathematical means without emphasizing the phenomenological features, which are always of considerable speculative origin. Moreover, the methods of this work can be easily applied to the majority of inflationary scenarios.

I would also like to point that in supersymmetry breaking terms SIDT is of the unbroken type. Since several years, Witten questions this fundamental issue [13]. Most recently, he remarked [14]: "In the standard framework of low energy physics, *there appears to be no natural explanation for vanishing or extreme smallness of the vacuum energy*, while on the other hand it is very difficult to modify this framework in a sensible way. In seeking to resolve this problem, *one naturally wonders if the real world can somehow be interpreted in terms of a vacuum state with unbroken supersymmetry*". In the cosmological context of this work, *there appears to be a simple technical explanation for the vanishing of the vacuum energy*, because only in this case eq. (4) reduces to a Riccati equation wherefrom one can proceed with SIDT, which is naturally of unbroken type and moreover it might be used to explain the supernovae Ia data as I argued herein. In the vacuum case, the supersymmetric one-parameter adiabatic indices are $\gamma_{\kappa,\mathrm{vac}}(\eta;\lambda) = -\frac{4}{3\kappa}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\Big[\ln(\mathcal{I}_\kappa(\mathrm{c}\eta)+\lambda)\Big]$, leading to plots similar to the other parametric barotropic fluids. However, for the vacuum $\mathrm{c} \leq 0$ holds ($\mathrm{c}_{\mathrm{vac}} = -1$). These cases are left for further study.

Finally, I mention that in the framework of supersymmetric quantum mechanics I published a direct iteration of this type of SIDT leading to multiple parameter 'zero modes' [15], which can be adapted to cosmology if the approach presented here is proven relevant.

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Plots

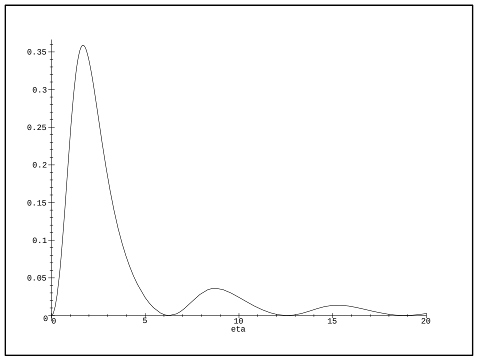

Fig. 1a. Zero mode scale factor of matter dominated universe for $\kappa = 1$ and $\lambda = 0.9$.

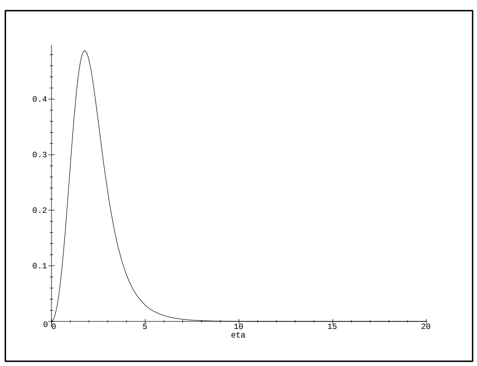

Fig. 1b. Zero mode scale factor of matter dominated universe for $\kappa = -1$ and $\lambda = 0.9$.

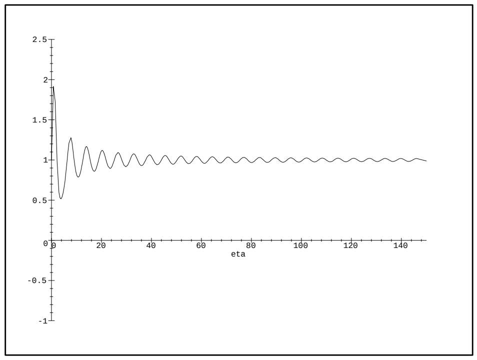

Fig. 1c. $\gamma_\kappa(\eta;\lambda)$ for a matter dominated universe of $\kappa=+1$ and $\lambda=0.9$.

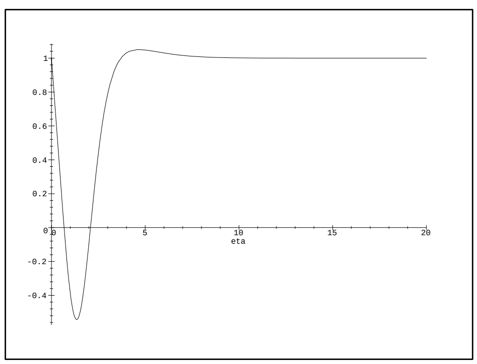

Fig. 1d. $\gamma_\kappa(\eta;\lambda)$ for a matter dominated universe of $\kappa=-1$ and $\lambda=0.9$.

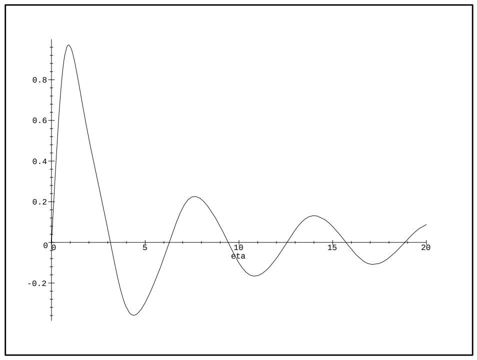

Fig. 2a. Zero mode scale factor of a radiation-filled universe for $\kappa = +1$ and $\lambda = 0.6$.

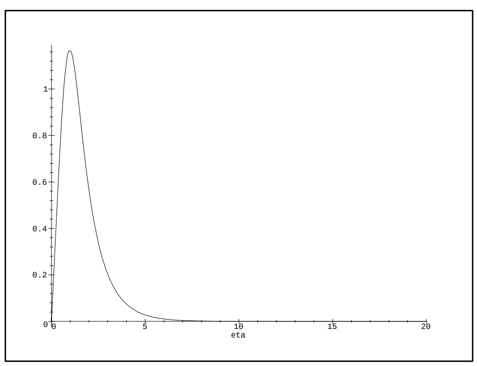

Fig. 2b. Zero mode scale factor of a radiation-filled universe for $\kappa = -1$ and $\lambda = 0.6$.

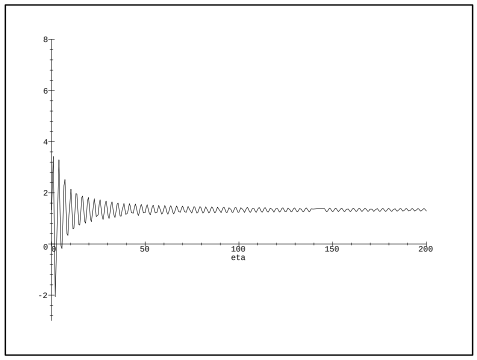

Fig. 2c. $\gamma_\kappa(\eta;\lambda)$ for a radiation dominated universe of $\kappa = +1$ and $\lambda = 0.6$.

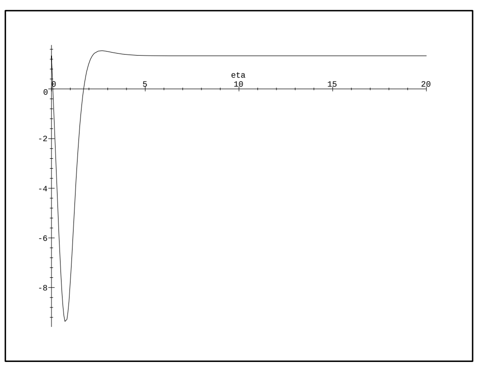

Fig. 2d. $\gamma_\kappa(\eta;\lambda)$ for a radiation dominated universe of $\kappa = -1$ and $\lambda = 0.6$.

Plots not included in the submission of 3/28/00

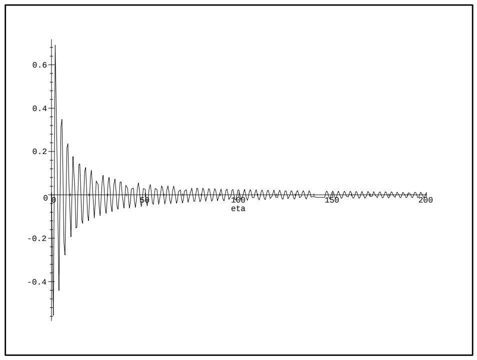

Fig. vac1. $\gamma_\kappa(\eta;\lambda)$ for a vacuum-dominated universe of $\kappa=1$ and $\lambda=0.9$.

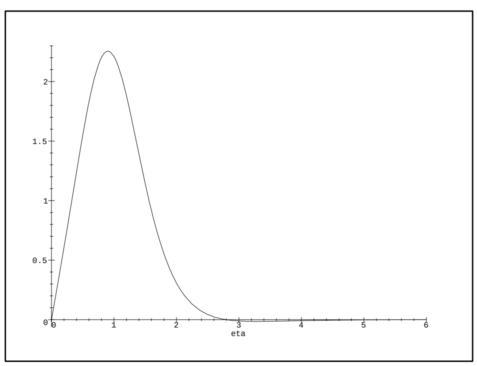

Fig. vac2. $\gamma_\kappa(\eta;\lambda)$ for a vacuum-dominated universe of $\kappa=-1$ and $\lambda=0.9$.